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Finite-Difference Solutions of a Non-linear Schrödinger Equation*

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This paper describes a finite-difference method to approximate a Schrödinger equation with a power non-linearity. A special case of this equation is currently used to model the propagation of a laser beam in a plasma. The main feature of the method we present is that it satisfies a discrete analogue of some important conservation laws of the equations. We present numerical results which show in particular the propagation and the formation of solitons in the one-dimensional case.

1. INTRODUCTION

We consider in $\mathbb{R}^n$, a complex-valued function $u(x, t)$ solution of the following non-linear Schrödinger equation.

$$\begin{gathered} i\frac{\partial u}{\partial t} - \Delta u + \lambda\,|u|^{p-1}\,u = 0, \\ u(x, 0) = \phi(x), \qquad p > 1,\ \lambda \text{ real}, \end{gathered} \tag{1.1}$$

where $\phi(x)$ is a sufficiently smooth function. This equation has been extensively studied in the past few years. For the sake of completeness we rapidly recall the main results concerning existence of solutions to Eq. (1.1). (We refer the reader to Strauss [8], where many important results are stated and sketch of most proofs is given.)

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Multiplying (1.1) by $\bar{u}$ and taking the imaginary part, then multiplying by $\partial\bar{u}/\partial t$ and taking the real part, one easily obtain two standard conservation laws, namely,

$$\int_{\mathbb{R}^n} |u|^2 \, dx = \text{constant} \tag{1.2}$$

and

$$E(u) = \int_{\mathbb{R}^n} \left\{ \tfrac{1}{2} |\nabla u|^2 + \frac{\lambda}{p+1} |u|^{p+1} \right\} dx = \text{constant}. \tag{1.3}$$

This shows that the case of negative λ may present some problems, because E can remain constant with neither of $\int_{\mathbb{R}^n} \frac{1}{2} |\nabla u|^2 \, dx$ or $\int_{\mathbb{R}^n} |u|^{p+1} \, dx$ remaining bounded. Indeed the behaviour of solutions depends heavily on the sign of λ, the parameter p and the space dimension n.

(i) For $\lambda > 0$, $p < \infty$, solutions exist for all t;

(ii) For $\lambda > 0$, $n > 2$, $p < 1 + 4/(n-2)$ or $\lambda < 0$, $p < 1 + 4/n$ solutions exist for all t, with some regularity properties;

(iii) For $\lambda < 0$, $p \geqslant 1 + 4/n$, a smooth solution cannot exist for any t if $E(\phi) < 0$.

Case (iii) provides a precise initial condition for the self-focusing of a laser beam. Finally let us recall that for $\lambda < 0$, $n = 1$ there exists a four-parameter family of solitary wave solutions

$$\begin{aligned} u(x, t) &= f(x - ct) \exp(ig(x - bt)), \\ f^{p-1}(x) &= -\frac{(p+1)\alpha}{2\lambda} \operatorname{sech}^2 \left[\frac{p-1}{2} \sqrt{\alpha}\,(x - x_c) \right], \\ g(x) &= \frac{c}{2} x + \phi, \qquad \alpha = \frac{c}{2} \left[\frac{c}{2} - b \right] > 0. \end{aligned} \tag{1.4}$$

The case $n = 1$, $p = 3$, $\lambda < 0$ has been studied in detail by Zakharov and Shabat [12]. Solitary waves of (1.4) are called *solitons* in this case and they can be written as

$$\begin{aligned} u(x, t) &= 2\eta f \exp\{ig\}, \\ f &= \operatorname{sech}(2\eta(x - x_c) + 8\eta\xi t), \\ g &= -2\xi x - 4(\xi^2 - \eta^2)t + \phi, \end{aligned} \tag{1.5}$$

where η, ξ, ϕ, x_c are parameters.

From these points, one must observe that the behavior of the solutions is quite intricate. In order to get reliable numerical solutions, one should develop numerical schemes satisfying the basic conservation laws that are the cornerstone of the theory.

We shall develop, in the next section, a finite-difference scheme in which discrete analogues of conservation laws (1.2), (1.3) will be satisfied. It has long been known that discrete conservation laws are an important feature in computing *smooth* solutions of hyperbolic equations. For instance, Zabusky and Kruskal [11] have proposed a numerical method for the closely related Korteweg–de Vries (K–dV) equation in which a discrete invariant existed. The K–dV equation also has soliton solutions.

In the last few years, the treatment of Eq. (1.1) and related equations has been a much-studied topic. Spectral and pseudo-spectral methods have been very popular in this respect. Very efficient codes can then be obtained using FFT methods. Such methods have been developed by Fleck *et al.* [3] for the linear Maxwell equations and also by Lax *et al.* [6], where a FFT is allied with a predictor–corrector time marching scheme. We also refer the reader to a very interesting work by Abe and Inoue [1], where many finite-difference methods and a spectral method are compared for the computation of solutions to the K–dV equation. Fornberg and Whitham [4] use a pseudo-spectral method for the Schrödinger equation and results are also presented in Yuen and Ferguson [10]. A finite-difference method for the Schrödinger equation is described in Degtyarev and Krylov [2]. They consider the case of cylindrical coordinates with axial symmetry and a Lagrangian formulation leading to a hydrodynamic analogy. This method also displays conservation properties. A treatment of a related problem can also be found in Karamzin [5].

We are interested in expanding waves and the periodicity condition of spectral methods is not suited to our needs. We choose to use a finite-difference method in a variable domain.

2. Numerical Scheme

We shall first introduce a slightly more general form of Eq. (1.1) including a dissipation term and some eventual non-homogeneity in the propagation medium. Consider with $r=|x|$:

$$i\frac{\partial u}{\partial t}+ivu-\Delta u+\lambda u(|u|^{p+1}+\alpha r)=0, \tag{2.1}$$

$$u(x,0)=\phi(x).$$

Conservation laws (1.2) and (1.3) become

$$\int_{\mathbb{R}^n}|u|^2\,dx=\int_{\mathbb{R}^n}|\phi(x)|^2\,dx\cdot e^{-2\nu t}, \tag{2.2}$$

$$\int_{\mathbb{R}^n}\tfrac{1}{2}|\nabla u|^2+\frac{\lambda}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}\,dx+\alpha\int_{\mathbb{R}^n}r\,|u(x,t)|^2\,dx=\text{constant}, \tag{2.3}$$

where $r=|x|$.

We now consider a special finite-difference scheme. For simplicity we present only the one-dimensional case. Extension to a higher dimension involves only the treatment of the $-\Delta u$ term, a standard and straightforward procedure. Other terms do not contain derivatives and their treatment is the same in any dimension.

The treatment of non-linear terms is very similar to the one presented in Strauss and Vazquez [9] for the related Klein–Gordon equation. Our scheme is second order accurate in both space and time. It is an implicit scheme and not a leap-frog one, as is often the case in the treatment of hyperbolic equations. The reason for this is to obtain discrete invariants.

Let Δx and Δt be respectively the space and time discretization mesh sizes and let u_j^n denote the value of u at the jth node and at time $n\,\Delta t$. As we consider the whole space, j varies from $-\infty$ to $+\infty$ that is $j \in Z$, where Z is the set of all positive and negative integers.

We denote by u^n (without subscript) the vector $\{u_j^n, j \in Z\}$. Let u^n be given; we shall compute u^{n+1} by solving for $n \geqslant 0, j \in Z$

$$
\begin{aligned}
& i\,\frac{u_j^{n+1} - u_j^n}{\Delta t} + iv\,\frac{|u_j^{n+1}|^2 + |u_j^n|^2}{|u_j^{n+1}|^2 - |u_j^n|^2}\,(u_j^{n+1} - u_j^n) \\
&\quad - \frac{1}{2}\left\{\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{|\Delta x|^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{|\Delta x|^2}\right\} \\
&\quad + \frac{\lambda}{p+1}\left\{\frac{|u_j^{n+1}|^{p+1} - |u_j^n|^{p+1}}{|u_j^{n+1}|^2 - |u_j^n|^2} + aj\,\Delta x\right\}(u_j^{n+1} + u_j^n) = 0.
\end{aligned} \tag{2.4}
$$

The value of u_j^0 may be computed from $\phi(x)$, for instance, by

$$u_j^0 = \phi(j\,\Delta x)$$

or

$$u_j^0 = (1/\Delta x)\int_a^b \phi(x)\,dx, \qquad a = (j - \tfrac{1}{2})\,\Delta x, \qquad b = (j + \tfrac{1}{2})\,\Delta x \tag{2.5}$$

The only non-evident features of (2.4) are in the second and fourth terms. The reason for this strange way of discretizing such simple terms as ivu or $\lambda\,|u|^{p-1}\,u$ will become clear in the following and lies in our desire to obtain discrete conservation laws analogous to (2.2) and (2.3).

Remark 2.1. If p is an odd integer larger than 3, it is easy to check that our discretization of the non-linear terms is consistant with $\lambda\,|u|^{p-1}\,u$ and indeed simple computational forms can be derived. In the important particular case $p = 3$, we get the obvious formula.

$$\left\{\frac{|u_j^{n+1}|^2 + |u_j^n|^2}{2} + aj\,\Delta x\right\}\frac{(u_j^{n+1} + u_j^n)}{2} \tag{2.6}$$

as the fourth term of (2.4). We now show that this scheme satisfies the discrete analogues of (2.2), (2.3).

PROPOSITION 2.1. *For all* $n \geqslant 0$,

$$\Delta x \sum_{j \in Z} |u_j^n|^2 = \left\{ \frac{1 - \nu \Delta t}{1 + \nu \Delta t} \right\}^n \Delta x \sum_j |u_j^0|^2, \tag{2.7}$$

$$\frac{\Delta x}{2} \sum_j \left| \frac{u_{j+1}^n - u_j^n}{\Delta x} \right|^2 + \frac{\lambda}{p+1} \Delta x \sum_j |u_j^n|^{p+1} + \frac{\lambda}{2} \alpha \Delta x \sum_j j |u_j^n|^2$$
$$= \text{constant} \tag{2.8}$$

Proof. In order to get (2.7) one multiplies (2.4) by $\bar{u}_j^{n+1} + \bar{u}_j^n$, where $\bar{u}$ is the complex conjugate of u. One sums over j and takes the imaginary part. Explicitly, the first term gives

$$\begin{aligned} i[\bar{u}_j^{n+1} + \bar{u}_j^n] \frac{u_j^{n+1} - u_j^n}{\Delta t} &= \frac{i}{\Delta t} [\bar{u}_j^{n+1} - u_j^{n+1} - \bar{u}_j^n u_j^n + \bar{u}_j^n u_j^{n+1} - \bar{u}_j^{n+1} u_j^n] \\ &= \frac{i}{\Delta t} [|u_j^{n+1}|^2 - |u_j^n|^2 + 2i \operatorname{Im}(\bar{u}_j^n u_j^{n+1})]. \end{aligned} \tag{2.9}$$

and the second term

$$\begin{aligned} & i\nu \left| \frac{|u_j^{n+1}|^2 + |u_j^n|^2}{|u_j^{n+1}|^2 - |u_j^n|^2} \right| (u_j^{n+1} - u_j^n)(\bar{u}_j^{n+1} + \bar{u}_j^n) \\ &= i\nu \left| \frac{|u_j^{n+1}|^2 + |u_j^n|^2}{|u_j^{n+1}|^2 - |u_j^n|^2} \right| \{|u_j^{n+1}|^2 - |u_j^n|^2 + 2i \operatorname{Im}(\bar{u}_j^n u_j^{n+1})\}. \end{aligned} \tag{2.10}$$

Taking the sum over j and keeping only the imaginary part, one obtains

$$\sum_j \left\{ \frac{1}{\Delta t} (u_j^{n+1}|^2 - |u_j^n|^2) + \nu (|u_j^{n+1}|^2 + |u_j^n|^2) \right\} = 0,$$

which is nothing but (2.7), for one can easily check that the last two terms are real.

In order to obtain the second conservation law, one multiplies (2.4) by $\bar{u}_j^{n+1} - \bar{u}_j^n$, sums over j and take the real part.

The first two terms are purely imaginary and therefore drop out of the sum. The third term (multiplied by $2(\Delta x)^2$) becomes.

$$\sum_j (\bar{u}_j^{n+1} - \bar{u}_j^n)[u_{j+1}^{n+1} - u_j^{n+1} + u_{j+1}^n - u_j^n] - \sum_j (\bar{u}_j^{n+1} - \bar{u}_j^n)[u_j^{n+1} - u_{j-1}^{n+1} + u_j^n - u_{j-1}^n]. \tag{2.11}$$

As the sum ranges over Z, one may change j to $j+1$ in the second term of (2.11) to obtain

$$
\begin{aligned}
&= \sum_j (\bar{u}_j^{n+1} - \bar{u}_j^n)[u_{j+1}^{n+1} - u_j^{n+1} + u_{j+1}^n - u_j^n] \\
&\quad - \sum_j (\bar{u}_{j+1}^{n+1} - \bar{u}_{j+1}^n)[u_{j+1}^{n+1} - u_j^{n+1} + u_{j+1}^n - u_j^n] \\
&= -\sum_j [\bar{u}_{j+1}^{n+1} - \bar{u}_j^{n+1} - (\bar{u}_{j+1}^n - \bar{u}_j^n)][u_{j+1}^{n+1} - u_j^{n+1} + u_{j+1}^n - u_j^n] \qquad (2.12) \\
&= \sum_j \{|u_{j+1}^n - u_j^n|^2 - |u_{j+1}^{n+1} - u_j^{n+1}|^2 + 2i \operatorname{Im}[(\bar{u}_{j+1}^n - \bar{u}_j^n)(u_{j+1}^{n+1} - u_j^{n+1})]\}.
\end{aligned}
$$

Taking the real part, this becomes

$$
\frac{1}{2|\Delta x|^2} \sum_j |u_{j+1}^n - u_j^n|^2 - |u_{j+1}^{n+1} - u_j^{n+1}|^2. \qquad (2.13)
$$

The last term gives

$$
\begin{aligned}
&\frac{\lambda}{p+1} \sum_j \left\{ \frac{|u_j^{n+1}|^{p+1} - |u_j^n|^{p+1}}{|u_j^{n+1}|^2 - |u_j^n|^2} + \alpha j \Delta x \right\} (\bar{u}_j^{n+1} - \bar{u}_j^n)(u_j^{n+1} + u_j^n) \\
&\quad = \frac{\lambda}{p+1} \sum_j \left\{ \frac{|u_j^{n+1}|^{p+1} - |u_j^n|^{p+1}}{|u_j^{n+1}|^2 - |u_j^n|^2} + \alpha j \Delta x \right\} \\
&\quad \times (|u_j^{n+1}|^2 - |u_j^n|^2 + 2i \operatorname{Im}(\bar{u}_j^{n+1} u_j^n),
\end{aligned}
$$

and its real part provides the last term in (2.8).

Remark 2.2. For $\nu = 0$, (2.7) is indeed strictly a conservation law. For $\nu > 0$ we have a second order approximation to the continuous case of (2.2).

Remark 2.3. We have presented a finite-difference method, but it can easily be seen that the previous development can easily be extended to the case where the space discretization is done through standard conforming finite elements. Let us just consider again the one-dimensional case and let u_h^n be a discrete solution belonging to V_h, a space of continuous piecewise polynomial functions. u_h^n at time $n \Delta t$, we compute u_h^{n+1} as solution of

$$
\begin{aligned}
&i \int_{\mathbb{R}} (u_h^{n+1} - u_h^n)\, v_h\, dx + i\nu \int_{\mathbb{R}} \frac{|u_h^{n+1}|^2 + |u_h^n|^2}{|u_h^{n+1}| - |u_h^n|^2} (u_h^{n+1} - u_h^n)\, v_h\, dx + \cdots \\
&\quad + \int_{\mathbb{R}} \nabla(u_h^{n+1} + u_h^n) \cdot \nabla v_h\, dx + \frac{\lambda}{p+1} \\
&\quad \times \int_{\mathbb{R}} \frac{|u_n^{n+1}|^{p+1} - |u_h^n|^{p+1}}{|u_h^{n+1}| - |u_h^n|^2} (u_h^{n+1} + u_h^n)\, v_h\, dx + \cdots \\
&\quad + \frac{\lambda \alpha}{2} \int_{\mathbb{R}} x(u_h^{n+1} + u_h^n)\, v_h\, dx = 0. \qquad (2.14)
\end{aligned}
$$

It can easily be seen, using the same technique as above, that from (2.14) one has

$$\int_{\mathbb{R}} |u_h^n|^2 \, dx = \left[\frac{1-\nu \Delta t}{1+\nu \Delta t}\right]^n \int_{\mathbb{R}} |\phi(x)|^2 \, dx, \tag{2.15}$$

$$\frac{1}{2}\int_{\mathbb{R}} |\nabla u_h^n|^2 \, dx + \frac{\lambda}{p+1}\int_{\mathbb{R}} |u_h^n|^{p+1} + \frac{\lambda\alpha}{2}\int_{\mathbb{R}} r\,|u_h^n|^2 \, dx = \text{constant.} \tag{2.16}$$

Of course (2.14) induces for a regular mesh a finite difference scheme which is slightly more complex than ours.

It is moreover highly probable that discrete conservation laws such as (2.16) and (2.17) could provide a firm basis to get a convergence proof for a scheme like (2.14). An advantage of finite elements is that one can easily obtain higher order approximations by using appropriate polynomials. ∎

3. Numerical Solution of the Discrete Equations

In order to compute a solution, we must first make an additional hypothesis about solutions, for it is impossible to compute on the whole real line. We must therefore suppose that our solution has a compact support and that it is zero outside some interval $[x_0, x_{N+1}]$. We use the artificial boundary conditions $u(x_0) = u(x_{N+1}) = 0$; it would be possible to use other boundary conditions if some were available. This hypothesis of compact support is not exact in general. However, many solutions decrease rapidly at infinity and it seems reasonable to use such an approximation. It must be noted that the computer program must provide a facility to vary $[x_0, x_{N+1}]$ during computations in order to follow traveling solutions like solitons.

We now write our system of non-linear equations in "matrix" form. Let A be the matrix associated with the discrete Laplacian operator. In the simple one-dimension case of (2.4), A is tridiagonal but this is not the case for higher dimensions or higher-degree approximations. Let X denote the diagonal matrix, where the jth diagonal entry is $|x_j| = r_j$, where x_j is the (vector) coordinate of the jth node. For $u^n \in \mathbb{C}^n$ given, we define a mapping $F_n(u)$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ of which the jth component is given by.

$$\begin{aligned}(F_n(u))_j &= \frac{\lambda}{p+1}\left(\frac{|u_j|^{p+1} - |u_j^n|^{p+1}}{|u_j|^2 - |u_j^n|^{p+1}}\right)(u+u^n) && \text{if} \quad u_j \neq u_j^n, \\ &= \lambda\,|u_j|^{p-1}\,u_j && \text{if} \quad u_j = u_j^n,\end{aligned} \tag{3.1}$$

and similarly $G_n(u)$ with

$$\begin{aligned}(G_n(u))_j &= i\nu\frac{|u_j|^2 + |u_j^n|^2}{|u_j|^2 - |u_j^n|^2}(u_j - u_j^n) && \text{if} \quad u_j \neq u_j^n, \\ &= i\nu u_j && \text{if} \quad u_j = u_j^n.\end{aligned} \tag{3.2}$$

We can now rewrite our system (2.4) in the more compact form.

$$\frac{i}{\Delta t}(u^{n+1}-u^n)+(A+\lambda\alpha X)\left(\frac{u^{n+1}+u^n}{2}\right)+G_n(u^{n+1})+F_n(u^{n+1})=0, \quad (3.3)$$

or equivalently,

$$\left[\frac{i}{\Delta t}I+\frac{A}{2}+\frac{\lambda\alpha}{2}X\right]u^{n+1}+F_n(u^{n+1})+G_n(u^{n+1})=\left[\frac{i}{\Delta t}I-\frac{A}{2}-\frac{\lambda\alpha}{2}X\right]u^n. \quad (3.4)$$

This is a non-linear system and it must be solved by some iterative technique. Although other possibilities should be explored, we obtained a very simple successive approximation method.

Let $u_0=u^n$ be given, we compute a sequence $\{u_p\}_{p=0,1\ldots}$ of vectors of $\mathbb{C}^n$, by the inductive relation

$$u_{p+1}=\left[\frac{i}{\Delta t}+\frac{A}{2}+\frac{\lambda\alpha}{2}X\right]^{-1}\left[\left(\frac{i}{\Delta t}I-\frac{A}{2}-\frac{\lambda\alpha}{2}X\right)u^n-F_n(u_p)\,G_n(u_p)\right]. \quad (3.5)$$

This sequence is easily computed for it implies only the solution of a linear system that can be factorized once and for all. As is standard in similar cases (3.5) proved to be convergent when Δt is small enough.

As to the stopping test we found that a very good one was to check on u^{n+1} for the conservation laws (2.16)–(2.17) that are to be verified if u^{n+1} has been computed with sufficient accuracy.

4. Numerical Results

We present in this section a few illustrative examples of results obtained by the method just described. We consider

(A) Propagation of a soliton with or without dissipation ($\nu=0$ or $\nu=0.1$).

(B) Interaction of two colliding solitons.

(C) Emergence of a soliton from a square well initial condition.

All computations were done for $0\leqslant t\leqslant 6$, with a time step $\Delta t=0.02$ and $\Delta x=0.1$. In examples A and B we had $-30<x<30$ and in example C, $-70\leqslant x\leqslant 70$. We present perspective views of the results; a curve is drawn at every 10th time step.

Example A. Propagation of a soliton.

We used 601 points in $[-30,+30]$. We define

$$u(x_j,0)=2\eta\exp\{(i(-2\xi x_j+\phi)\}\operatorname{sech}\{2\eta(x-x_c)\}, \quad (4.1)$$

with $\eta=0.75$, $\xi=-1$, $\phi=0$, $x_c=-5$.

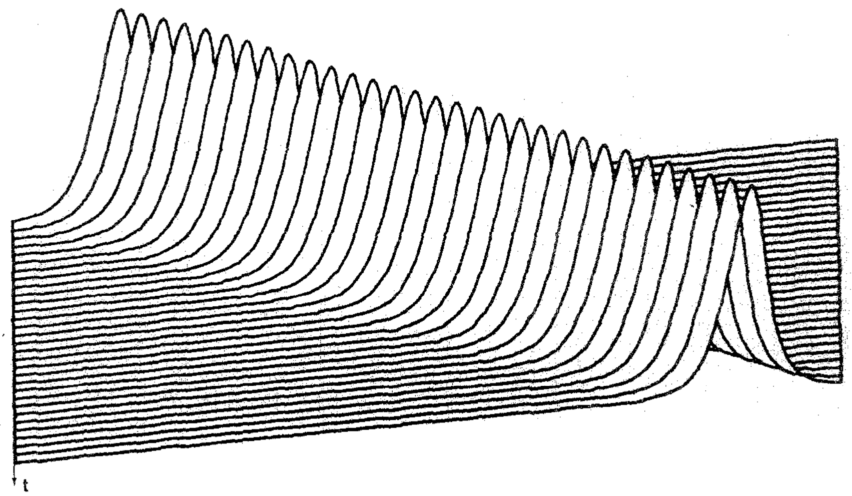

FIG. 1. Propagation of a single soliton ($\nu = 0$).

This is the discretization at time $t = 0$ of a soliton with amplitude $2\eta = 1.5$, traveling with a velocity $-4\xi = +4$ and initially centered at $x_c = -5$. Using (4.1) the discrete invariants defined by (2.7) and (2.8) are respectively 3 and 0.954713. These values must remain constant for $\nu = 0$. For $\nu > 0$ the first one must decay. The result for $\nu = 0$ is presented in Fig. 1, which presents a perspective view of the traveling soliton from $t = 0$ to $t = 6$. Invariants have been kept to four digits. With $\nu = 0.1$, there is a strong dissipative term. The result is presented in Fig. 2. The soliton decays in amplitude. Small ripples begin to appear in the curves for $t > 2$. They are caused by a lack of conservation of the theoretical invariants; at $t = 6$, the second invariant

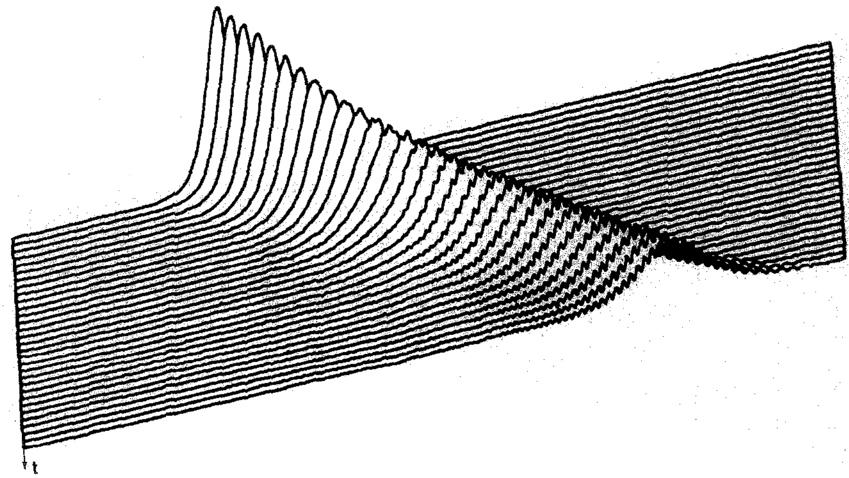

FIG. 2. Propagation of a single soliton ($\nu = 0.1$).

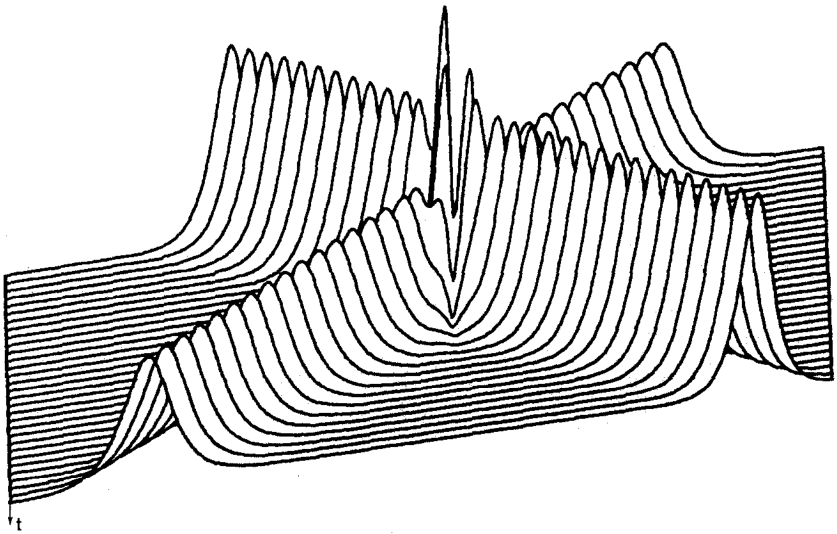

FIG. 3. Intersection of two colliding solitons.

(2.8) has decreased to ≈ 0.85 from an initial value of 0.954. This phenomenon comes from the difficulties involved in the numerical handling of the term

$$iv \frac{(|u_j^{n+1}|^2 + |u_j^n|^2}{|u_j^{n+1}|^1 - |u_j^{n2}|^2} (u_j^{n+1} - u_j^n),$$

which contains the difference of nearly equal quantities. Other ways of treating the dissipative term should therefore be explored. This result clearly shows the need of good conservation properties to get reliable results.

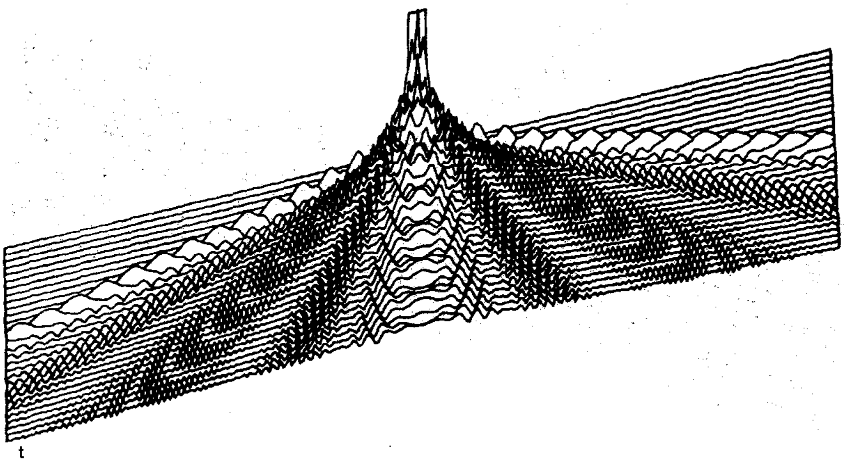

FIG. 4. Diffusion of initial square wave.

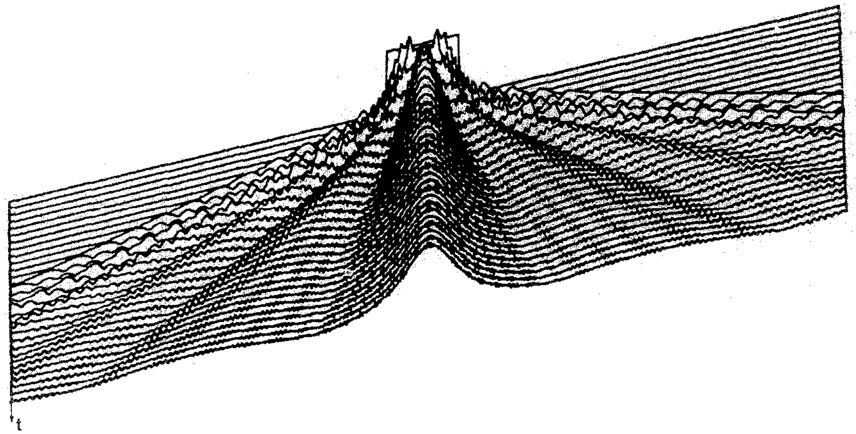

FIG. 5. Birth of a soliton.

EXAMPLE B: Interaction of two solitons.
We used 601 mesh points in $[-30, +30]$ at $t=0$.
We define

$$u(x_j, 0) = s_1(x_j, 0) + s_2(x_j, 0), \tag{4.2}$$

where s_1 and s_2 are two solitons as in (4.1) with respective amplitude $2\eta_1 = 1$ and $2\eta_2 = 1.5$ and $\xi_1 = -1$; $\xi_2 = +1$; $x_{c_1} = -15$, $x_{c_2} = +5$. These two solitons, traveling in opposite directions, collide and separate, conserving their initial shape. The numerical results are presented in Fig. 3. Invariant were conserved to five significant digits.

EXAMPLE C: Birth of a soliton.
Figures 4 and 5 present the results obtained from respectively

$$\begin{aligned} u(x, 0) &= 1.2 \qquad \text{if} \quad -.6 \leqslant x \leqslant +.6, \\ &= 0 \qquad \text{elsewhere}, \end{aligned} \tag{4.3}$$

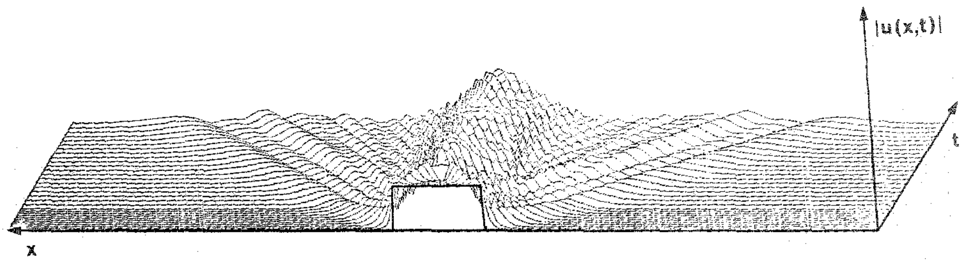

FIG. 6. Birth of a soliton (rear view).

and

$$u(x, 0) = 0.52 \qquad \text{if} \quad -2.6 \leqslant x \leqslant 2.6$$
$$= 0 \qquad \text{elsewhere.}$$

In the first case $\int_{-\infty}^{\infty} u(x, 0)\, dx$ is equal to 1.44 and in the second case to 2.70. Theory predicts that a soliton should appear for a value larger than $\pi/2$ (cf. Payre [4]). This is clearly seen in the numerical results: the solution of Fig. 5 fades out while that of Fig. 6 gives birth to a standing soliton. Figure 6 presents a rear view of the first steps of the computation in Fig. 5.

6. Conclusion

We developed a reliable finite difference method for the solution of non-linear Schrödinger equation. This method could be extended to related problems and the fact the discrete conservation laws are derived enables the user to easily check the validity of its results. A convergence proof however remains to be done.

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